HOMOLOGICAL EIGENVALUES OF MAPPING CLASSES AND TORSION HOMOLOGY GROWTH FOR FIBERED 3–MANIFOLDS

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ABSTRACT. Let S be an orientable surface with negative Euler characteristic, let $\psi \in \mathrm{Mod}(S)$ be a mapping class of S, and let T_ψ be the mapping torus of ψ. We study the action of lifts of ψ on the homology of finite covers of S via the torsion homology growth of towers of finite covers of T_ψ. We show that ψ admits a lift to a finite cover with a homological eigenvalue of length greater than one if and only if the mapping torus T_ψ admits a finite cover X and a certain tower of abelian covers which have exponential torsion homology growth. We show that the existence of such a lift of ψ is intrinsic to T_ψ, in the sense that it does not depend on the particular fibration used to present T_ψ.

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1. Introduction

1.1. **Overview.** Let S be an orientable surface of genus g with n punctures, such that

$$\chi(S) = 2 - 2g - n < 0.$$

We will assume that S has no boundary components, and that the genus and number of punctures of S are finite. The **mapping class group** of S, denoted by $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation–preserving homeomorphisms of S.

The purpose of this article is to study the actions of an individual mapping class $\psi \in \mathrm{Mod}(S)$ on the homology of finite covers of S to which ψ lifts. We are particularly interested in the case where ψ is pseudo-Anosov. The expansion factor of ψ will be written $\lambda(\psi)$. Then, $\log \lambda(\psi) = h(\psi)$ is the minimal topological entropy of a representative of ψ. We will use the notation $\widetilde{S}$ for a finite Galois cover of S to which ψ lifts, and $\widetilde{\psi}$ for a lift of ψ to $\widetilde{S}$. The choice of $\widetilde{\psi}$ is well–defined up to pre– and post–composition with an element of the Galois group of $\widetilde{S}$ over S.

When considering the action of the lifts of ψ on the homology of covers, we will write

$$\rho(\widetilde{\psi}_*) = \rho(\widetilde{\psi}_* \mid H_1(\widetilde{S}, \mathbb{C})),$$

where the right hand side denotes the spectral radius of the action of $\widetilde{\psi}$ on $H_1(\widetilde{S}, \mathbb{C})$.

1.2. **Surface conjectures.** Our point of departure is the following pair of conjectures, both of which are currently open.

Conjecture 1.1 (Surface conjectures). *Let $\psi \in \mathrm{Mod}(S)$ be pseudo-Anosov. Then:*

- **(S1)** *There exists a lift $\widetilde{\psi}$ whose action on $H_1(\widetilde{S}, \mathbb{C})$ has infinite order.*

In fact:

- **(S2)** *There exists a lift $\widetilde{\psi}$ whose action on $H_1(\widetilde{S}, \mathbb{C})$ satisfies $\rho(\widetilde{\psi}_*) > 1$.*

Clearly, (S2) implies (S1). The surface conjectures can be motivated by the relationship between homology and entropy. If S is a surface and ψ is a pseudo-Anosov homeomorphism with expansion factor λ, then there is an inequality

$$\rho(\psi_*) \leq \lambda(\psi).$$

The right–hand side of the inequality remains constant under passage to finite covers, but the left–hand side can potentially increase under taking covers. In this article, we wish to understand how much the left–hand side can increase.

The following result of McMullen ([17]) shows that the gap in the inequality above may persist over all finite covers. Thus, a strengthening of (S2) which would posit that the supremum of $\rho(\widetilde{\psi}_*)$ over finite covers is equal to $\lambda(\psi)$ is generally false.

Theorem 1.2 (C. McMullen)**.** *For any pseudo-Anosov ψ, either:*

(1) *(Special case): there is a two–fold lift $\widetilde{\psi}$ such that*
$$\rho(\widetilde{\psi}_*) = \lambda(\psi),$$
or

(2) *(General case): we have*
$$\sup \rho(\widetilde{\psi}_*) < \lambda(\psi),$$
where the supremum is taken over all lifts of ψ to finite covers $\widetilde{S} \to S$.

The following result, which is proved by the author in [8], gives some evidence for (S1):

Theorem 1.3. *For each nontrivial mapping class ψ, there is a finite cover $\widetilde{S}$ of S such that each lift of ψ to $\mathrm{Mod}(\widetilde{S})$ acts nontrivially on $H_1(\widetilde{S}, \mathbb{C})$.*

1.3. **Torsion conjectures.** To further give evidence for conjectures (S1) and (S2), we relate these statements to well–known conjectures in the theory of 3–manifolds. Given $\psi \in \mathrm{Mod}(S)$, we let
$$T_\psi = S \times I/((s,0) \sim (\psi(s),1))$$
be the mapping torus of ψ. The fundamental group Γ of T_ψ is the semidirect product of $\pi_1(S)$ and $\mathbb{Z}$, with the generator $1 \in \mathbb{Z}$ acting by ψ. The isomorphism type of Γ is independent of the lift of ψ to $\mathrm{Aut}(\pi_1(S))$ (see for instance [2], Chapter 2, Prop. 12). It is well–known by a theorem of Thurston that T_ψ is hyperbolic (it admits a complete metric of constant negative sectional curvature equal to -1) if and only if ψ is pseudo-Anosov.

A **tower** of finite covers of T is a sequence of connected finite covers
$$\cdots \to T_3 \to T_2 \to T_1 \to T.$$
We require $\deg(T_i \to T) \to \infty$ as $i \to \infty$. A sequence of finite index subgroups
$$\cdots \leq \pi_1(T_2) \leq \pi_1(T_1) \leq \pi_1(T)$$
with $[\pi_1(T) : \pi_1(T_i)] \to \infty$ as $i \to \infty$ determines a tower of finite covers of T. A tower so determined is called **exhausting** if
$$\bigcap_i \pi_1(T_i) = \{1\}.$$

The following conjectures are suggested by the work of Bergeron–Venkatesh in [3] and of Lück in [13] and [14].

Conjecture 1.4 (Torsion conjectures)**.** *Let $T = T_\psi$ be the mapping torus of a pseudo-Anosov homeomorphism $\psi \in \mathrm{Mod}(S)$. Then:*

- **(T1)** *The manifold T has exponential torsion homology growth with respect to some tower of finite covers $\{T_i\}$ of T.*

In fact:

- **(T2)** *The manifold T has exponential torsion homology growth with respect to every tower of exhausting finite covers $\{T_i\}$ of T.*

Notice that conjecture (T1) does not require the tower to be exhausting. Conjecture (T2) is Lück's conjecture 1.12 in [14].

A 3–manifold T has **exponential torsion homology growth** with respect to a tower of finite covers $\{T_i\}$ if

$$\limsup_i \frac{\log |H_1(T_i, \mathbb{Z})_{tors}|}{[\deg(T_i \to T)]} > 0.$$

The value of this limit is called the **torsion homology growth exponent**.

We remark that torsion homology growth as described above is special to 3–manifolds. In higher dimensions, homology in dimensions other than one must also be considered. See [3] for a correct definition of torsion homology growth for higher dimensional locally symmetric spaces.

We will show that the torsion conjectures (T1) and (T2) are logically related to the surface conjectures (S1) and (S2) through the following two results:

Theorem 1.5. *Let ψ be pseudo-Anosov, and suppose T_ψ has exponential torsion homology growth with respect to every exhausting tower of finite covers. Then there is a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$.*

Theorem 1.6. *Let ψ be pseudo-Anosov, and suppose there exists a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$. Then T_ψ has exponential torsion homology growth with respect to some tower of finite covers.*

Thus for any pseudo-Anosov homeomorphism ψ, we have (T2) implies (S2) implies (T1). The proof of Theorem 1.5 is short enough to include here:

Proof of Theorem 1.5. Suppose that no such lift exists. Then for each lift of ψ to a finite cover, we have $\rho(\widetilde{\psi}_*) = 1$. Take any exhausting, nested sequence $\{S_i\}$ of finite covers of S to which ψ lifts. By assumption, $\rho(\widetilde{\psi}_*) = 1$ for each of these covers. For each i, there is an exponent $N_i > 0$ such that the action of a lift $\widetilde{\psi}_*^{N_i}$ on $H_1(S_i, \mathbb{C})$ has spectrum $\{1\}$. Since $\rho(\widetilde{\psi}_*^{N_i}) = 1$, we have that if $M_i = i \cdot N_i$ then $\rho(\widetilde{\psi}_*^{M_i}) = 1$ as well. Furthermore, $M_i \to \infty$ as $i \to \infty$. It follows that $H_1(T_{\widetilde{\psi}^{M_i}}, \mathbb{Z})$ is torsion–free. Indeed, any matrix representing $\widetilde{\psi}_*^{M_i}$ is conjugate to an upper triangular matrix with ones down the diagonal. The homology $H_1(T_{\widetilde{\psi}^{M_i}}, \mathbb{Z})$ is given by

$$\mathbb{Z} \oplus H_1(S_i, \mathbb{Z})/(\widetilde{\psi}_*^{M_i} - I),$$

which then has no torsion. Since the $\{S_i\}$ exhaust S and M_i tends to infinity, the covers $\{T_{\widetilde{\psi}^{M_i}}\}$ will exhaust T_ψ. □

The proof of Theorem 1.5 shows that if (S2) fails then (T2) also fails dramatically: there is an exhausting sequence of finite covers of T_ψ with no torsion homology.

We will discuss Theorem 1.5 in more depth and give a proof of Theorem 1.6 in Section 4. Growth of torsion homology will allow us to characterize mapping classes with the property that each finite lift $\widetilde{\psi}$ satisfies $\rho(\widetilde{\psi}_*) = 1$, intrinsically from the mapping torus. A precise theorem is:

Theorem 1.7. *Let T_ψ be the mapping torus of a pseudo-Anosov mapping class ψ. The following are equivalent:*

1. *The mapping class ψ has a lift $\widetilde{\psi}$ to a finite cover which satisfies $\rho(\widetilde{\psi}_*) > 1$.*
2. *There is a finite cover X of T_ψ such that X has exponential growth of torsion homology with respect to some universal tower of finite abelian covers.*

Here, a **universal tower of abelian covers** of a finite CW complex X is a tower $\{X_i\}$ of finite abelian covers of X with the property that if $X_A \to X$ is any finite abelian cover of X, then for all sufficiently large i, the cover $X_i \to X$ factors through $X_A \to X$. The **homology covers** of a finite CW complex X are the covers $\{X_N\}$ induced by the canonical maps

$$\pi_1(X) \to H_1(X, \mathbb{Z}/N\mathbb{Z}).$$

A sequence of homology covers $\{X_{N_i}\}$ is a universal tower of abelian covers if and only if $N_i \mid N_{i+1}$ for each i, and if for each positive integer d, we have that $d \mid N_i$ for all sufficiently large i.

We remark that there is a natural notion of an **exhausting tower of abelian covers**, which is related to the notion of a universal tower of abelian covers. An exhausting tower of abelian covers of a finite CW complex X is a tower of covers $\{X_i\}$ determined by a nested sequence of finite index subgroups $\{\Gamma_i\}$ of $\pi_1(X)$ which satisfy

$$\bigcap_i \Gamma_i = [\pi_1(X), \pi_1(X)].$$

In our proof of Theorem 1.7, it will not be enough to assume that the tower in part (2) is an exhausting tower of abelian covers – we will need that the tower is a universal tower of abelian covers.

Theorem 1.7 suggests a statement which fits in between torsion conjectures (T1) and (T2) above, and which is logically equivalent to surface conjecture (S2):

Conjecture 1.8 (Torsion Conjecture (T1.5))**.** *Let ψ be a pseudo-Anosov mapping class and let T_ψ be its mapping torus. Then there exists a finite cover X of T_ψ and a universal tower of finite abelian covers $\{X_i\}$ of X which have exponential torsion homology growth.*

To see that (T2) implies (T1.5) implies (T1), observe that first Theorem 1.5 shows that (T2) implies (S2). By Theorem 1.7, we have that (S2) and (T1.5) are logically equivalent. That (T1.5) implies (T1) is immediate.

Let $f \in \mathrm{Mod}(S)$ and $g \in \mathrm{Mod}(S')$ be mapping classes, perhaps of different surfaces. We will say that f and g are **commensurable** if the associated mapping tori T_f and T_g are commensurable, which is to say that T_f and T_g are both finitely covered by another 3–manifold $M_{f,g}$. Notice that it follows immediately from the definitions that if f and g are the monodromies associated to different fibrations of a given 3–manifold, then f and g are commensurable. Notice furthermore that if f and g are commensurable and if f is pseudo-Anosov, then g is automatically pseudo-Anosov.

The following result will be an easy consequence of the proof of Theorem 1.7:

Corollary 1.9. *Let ψ be a pseudo-Anosov mapping class. There exists a lift $\widetilde{\psi}$ to a finite cover satisfying $\rho(\widetilde{\psi}_*) > 1$ if and only if for each mapping class ϕ which is commensurable to ψ, there exists a lift $\widetilde{\phi}$ to a finite cover satisfying $\rho(\widetilde{\phi}_*) > 1$.*

1.4. **Notes and references.** The expansion factor for a pseudo-Anosov homeomorphism ψ is closely related to its topological entropy. If $\lambda(\psi)$ is the expansion factor then $h(\psi) = \log \lambda(\psi)$ is the minimal topological entropy of a representative in the isotopy class of ψ. For homeomorphisms of compact surfaces, the inequality

$$\log \rho(\psi) \leq h(\psi)$$

was established by Manning in [15]. For diffeomorphisms of more general compact manifolds, the same inequality is obtained by Yomdin in [23].

With regards to the torsion conjectures, some recent work includes that of Bergeron and Venkatesh in [3] and of Lück in [13] and [14]. Bergeron and Venkatesh establish positive torsion homology growth for congruence covers of arithmetic hyperbolic 3–manifolds, though their ring of coefficients is not $\mathbb{Z}$. They conjecture positive torsion homology growth for more general classes of locally symmetric spaces and more general rings of coefficients. As noted above, conjecture (T2) is Lück's conjecture 1.12 in [14].

For a finite–volume hyperbolic 3–manifold M, the exact value of the limit in torsion conjecture (T2) has been conjectured to be the volume of M divided by 6π. This conjecture was made by T. Le and is discussed in [3] as well. T. Le has proved that the limit cannot exceed $\mathrm{vol}(M)/6\pi$.

In [5], Farb, Leininger and Margalit study **short** dilatation pseudo-Anosov mapping classes, showing that they all come from fibrations of Dehn fillings on finitely many 3–manifolds. Here,

short means that one fixes a number $M > 1$ and considers the mapping classes on all surfaces which satisfy $\lambda(\psi)^{|\chi(S)|} \leq M$. In light of Corollary 1.9, it is interesting to ask whether the set of short dilatation pseudo-Anosov mapping classes fall into finitely many commensurability classes. The answer is probably no.

There are generalized notions of Mahler measure (see Section 3) which work for polynomials in several variables. By analyzing the Mahler measure of the multivariable Alexander polynomial of a 3–manifold M, one can understand torsion homology growth along towers of finite covers of M. This approach has been studied independently by T. Le and by J. Raimbault in [12] and [18], respectively. Their methods also give Theorem 1.7.

Combining the results of this article with those in [9], we obtain the following general result about the homology of finite covers of a fibered 3–manifold:

Theorem 1.10. *Let $\psi \in \mathrm{Mod}(S)$ and let T_ψ be the mapping torus of ψ. There exists a finite cover X of T_ψ and a tower $\{X_i\}$ of finite abelian covers of X such that at least one of the following holds:*

1. *The tower $\{X_i\}$ has exponential torsion homology growth.*
2. *We have*

$$\inf_i \frac{b_1(X_i)}{\deg(X_i \to X)} > 0.$$

Proof. If ψ admits a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$ then Theorem 1.7 gives conclusion (1). If no such lift exists, then a corollary of the main result of [9] shows that $\pi_1(T_\psi)$ virtually surjects to a nonabelian free group, whence conclusion (2) follows. □

By recent work of Agol, Groves and Manning in [1], it seems that conclusion (2) should always hold for any finite volume hyperbolic 3–manifold. The work here and in [9] is independent of [1].

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3. Background

3.1. **The classification of mapping classes.** Let $\psi \in \mathrm{Mod}(S)$. Then ψ has either finite order or infinite order in $\mathrm{Mod}(S)$. In the former case, ψ is called **finite order**.

Proposition 3.1. *Let ψ be a finite order mapping class, and suppose S admits at least one complete hyperbolic metric of finite volume. Then there exists a hyperbolic metric on S and a representative $\widetilde{\psi}$ in the homotopy class of ψ which is an isometry in that metric and hence has finite order as a homeomorphism of S.*

Henceforth in this section, assume that ψ has infinite order. We have that ψ acts on the set of homotopy classes of simple, essential, nonperipheral (puncture–parallel) closed curves on S. If ψ has a finite orbit in this set, we say that ψ is **reducible**. If ψ has no such finite orbits, we say that ψ is pseudo-Anosov. A mapping class is called **pure** if there is a (possibly empty) multicurve $\mathcal{C}$ such that ψ stabilizes each component of $S \setminus \mathcal{C}$ and each component of $\mathcal{C}$, and such that the restriction of ψ to any component of $S \setminus \mathcal{C}$ is either the identity or is a pseudo-Anosov homeomorphism. Each mapping class has a power which is pure.

Pseudo-Anosov homeomorphisms have many equivalent definitions. For instance, they stabilize a pair of transverse measured foliations $\mathcal{F}^{\pm}$ which they stretch and contract respectively by a real algebraic integer $\lambda > 1$, called the **expansion factor** of ψ.

Proposition 3.2. *The expansion factor of a pseudo-Anosov homeomorphism ψ coincides with its growth rate as an outer automorphism of $\pi_1(S)$.*

An interesting feature of pseudo-Anosov homeomorphisms is that if we take any nontrivial, nonperipheral $\gamma \in \pi_1(S)$, then
$$\lim_{n\to\infty} (\ell(\psi^n(\gamma)))^{1/n} = \lambda(\psi),$$
independently of the generating set for $\pi_1(S)$ and the choice of γ.

We can summarize the classification as follows:

Theorem 3.3. *Let $\psi \in \mathrm{Mod}(S)$.*

1. *The mapping class ψ is finite order if and only if some isotopy representative of ψ has finite order as an isometry of some complete, finite volume hyperbolic metric on S.*
2. *The mapping class ψ is reducible if and only if there is a multicurve $\mathcal{C} \subset S$ such that, up to isotopy, $\psi(\mathcal{C}) = \mathcal{C}$.*
3. *The mapping class ψ is pseudo-Anosov if and only if it is neither finite order nor reducible.*

Good references for the proof of this result are [4], [6], [7] and [22].

3.2. **McMullen's spectral gap theorem.** The reference for this subsection is [17]. Throughout this subsection, let ψ be a pseudo-Anosov homeomorphism. In the previous section, we mentioned a distinction between **homological** and **non–homological** pseudo-Anosov homeomorphisms. The former class is characterized by orientable stable and unstable foliations $\mathcal{F}^{\pm}$, and equivalently by the invariant quadratic differential q being globally the square of a one–form ω. The spectral radius of the action of ψ on $H_1(S, \mathbb{R})$ is equal to $\lambda(\psi)$.

It is occasionally true that ψ is non–homological but becomes homological after passing to a finite cover of S. This happens if and only if the foliations $\mathcal{F}^{\pm}$ have only even–order singularities except for possibly one–pronged singularities at the punctures.

If ψ is a non–homological pseudo-Anosov then the spectral radius of the action of ψ on $H_1(S, \mathbb{R})$ is strictly smaller than the expansion factor (see [10] and [11]).

McMullen proved the following result, which is also stated in the introduction:

Theorem 3.4. *Suppose ψ is non–homological on each finite cover of S. Then the supremum over lifts to finite covers*
$$\sup \rho(\widetilde{\psi}_*)$$
is strictly less than the expansion factor $\lambda(\psi)$.

3.3. **Fibered 3–manifolds.** Let M be a 3–manifold which fibers of the circle. This means that there is a fibration
$$S \to M \to S^1$$
with the monodromy given by a homeomorphism ψ of S. It is well–known and easy to show that the homotopy type of $M = M_\psi$ depends only on the mapping class of ψ.

A result of Thurston (see [16] for a thorough discussion) says that the manifold M admits a hyperbolic metric of finite volume if and only if the monodromy ψ is a pseudo-Anosov homeomorphism. If ψ has finite order then a finite cover of M is homeomorphic to $S \times S^1$, and if ψ is reducible then M contains at least one non–peripheral essential torus. In either of those cases, M is not hyperbolic.

Given ψ, one can easily write a presentation for the fundamental group $\pi_1(M)$. The fact that M fits into a fibration implies that the fundamental group of M fits into a short exact sequence
$$1 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 1.$$
We can lift ψ to an automorphism of $\pi_1(S)$ which we also call ψ. A presentation can be given as follows:
$$\pi_1(M) = \langle \pi_1(S), t \mid t\pi_1(S)t^{-1} = \psi(\pi_1(S))\rangle.$$

Similarly, it is easy to compute $H_1(M, \mathbb{Z})$. Clearly, the homology group is a quotient of $\mathbb{Z} \oplus H_1(S, \mathbb{Z})$. On the other hand, each commutator is in the kernel of the abelianization map. So for

each $d \in H_1(S,\mathbb{Z})$, the element $\psi(d) - d$ is in the kernel of the abelianization. If $C(\psi)$ denotes the span of these elements in $H_1(S,\mathbb{Z})$, we have

$$H_1(M,\mathbb{Z}) \cong \mathbb{Z} \oplus H_1(S,\mathbb{Z})/C(\psi).$$

For the remainder of this subsection, [21] is the canonical reference. Let M be a compact, irreducible, atoroidal 3–manifold whose boundary (if any) is a finite union of tori. For any compact surface,

$$S = \bigcup_{i=1}^{n} S_i,$$

write $\chi_-(S)$ for the sum of the absolute values of the Euler characteristics of the components of S which have negative Euler characteristic. If $\phi \in H^1(M,\mathbb{Z})$ is a cohomology class, write

$$||\phi||_T = \min \chi_-(S),$$

where $[S] \in H_2(M, \partial M, \mathbb{Z})$ is dual to ϕ. This integer is called the **Thurston norm** of ϕ.

Theorem 3.5. *The Thurston norm has the following properties:*

(1) *The Thurston norm extends to a continuous norm on $H^1(M,\mathbb{R})$.*
(2) *The unit ball in the Thurston norm is a finite–sided polyhedron whose vertices are rational.*

Suppose M fibers over the circle. Then the fibration $M \to S^1$ yields a homomorphism $\pi_1(M) \to \mathbb{Z}$, which is a cohomology class. Such a cohomology class is called **fibered**. The homology class of the fiber is dual to this cohomology class and is norm–minimizing.

Theorem 3.6. *The following are properties of a fibered cohomology class ϕ:*

(1) *The cohomology class ϕ is an integral point in the interior of a cone C over a top–dimensional face of the unit norm ball.*
(2) *Every primitive integral cohomology class in the interior of C is also a fibered cohomology class.*

The Thurston norm is only interesting from the point of view of inequivalent fibrations if the rank of $H^1(M,\mathbb{R})$ is at least two, for otherwise there is only one primitive cohomology class in the cone over a top–dimensional face.

3.4. **Mahler measure.** The notion of Mahler measure will be useful in studying the growth of torsion homology. Let $p(t) \in \mathbb{Z}[t^{\pm 1}]$ be a monic Laurent polynomial. The **Mahler measure** of p is written

$$M(p) = \prod_i \max\{1, |\alpha_i|\},$$

where $\{\alpha_i\}$ ranges over the roots of p. If M is a square matrix over $\mathbb{C}$, we will write $\det{}'(M)$ for the **restricted determinant of** M, which is to say the product of the nonzero eigenvalues of M. We define the empty product to be equal to 1.

A general fact about Mahler measures is the following:

Lemma 3.7. *Let A be a square matrix with characteristic polynomial p_A. Then*

$$\lim_{n\to\infty} \frac{\log|\det{}'(A^n - I)|}{n} = \log M(p_A).$$

The relationship between Mahler measure and the size of $H_1(M,\mathbb{Z})_{tor}$ is thus given by the following:

Lemma 3.8. *Let $A \in GL_n(\mathbb{Z})$ and let*

$$1 \to \mathbb{Z}^n \to \Gamma \to \mathbb{Z} \to 1$$

be a semidirect product of $\mathbb{Z}$ and $\mathbb{Z}^n$ whose $\mathbb{Z}$–action is given by A. Then

$$|H_1(\Gamma, \mathbb{Z})_{tor}| = |\det{}'(A - I)|.$$

We will need the following elementary linear algebra result about the relationship between torsion homology growth, $\det{}'$ and matrices:

Lemma 3.9. *Let $A \in GL_n(\mathbb{C})$ have all eigenvalues on the unit circle. We have the inequality*

$$\log|\det{}'(A - I)| \leq n \cdot \log 2.$$

Proof. Viewing A as an automorphism of $\mathbb{C}^n$, we may conjugate A to express it as an upper triangular matrix. Each eigenvalue of μ of A satisfies $|\mu| = 1$, so that $|\mu - 1| \leq 2$. The conclusion of the lemma is immediate. □

Note that if A has all of its eigenvalues on the unit circle then so does A^N for each N. Notice furthermore that the bound given in Lemma 3.9 is sharp and is realized by $-I$, which in the world of mapping classes is realized by the hyperelliptic involution.

3.5. **Dependence on the lift.** Let $\psi \in \mathrm{Mod}(S)$ and let $\widetilde{S}$ be a finite Galois cover of S with Galois group G, and suppose ψ lifts to a mapping class $\widetilde{\psi}$ of $\widetilde{S}$. There is inherent ambiguity in the choice of the lift. Indeed, ψ is only an outer automorphism of $\pi_1(S)$ and of G. The fundamental group $\pi(S)$ acts on $\widetilde{S}$ via the Galois group G. The quotient of $\widetilde{S}$ by G is S. Thus, if $\widetilde{\psi}$ is pre–composed or post–composed with an element of G, the resulting homeomorphism of $\widetilde{S}$ is still a lift of ψ. We can write down all lifts of ψ by choosing an arbitrary lift $\widetilde{\psi}$ and then considering the elements

$$\{g \cdot \widetilde{\psi} \cdot h\},$$

where $g, h \in G$.

We thus obtain a natural map

$$\mathrm{Mod}(S) \to G \cdot \mathrm{Mod}(\widetilde{S}) \cdot G$$

which sends a mapping class of S to its set of lifts. The set of lifts of a mapping class ψ to $\mathrm{Mod}(\widetilde{S})$ is therefore the double coset $G\widetilde{\psi}G$.

Proposition 3.10. *Let $\widetilde{S}$ be a finite, Galois cover of S to which ψ lifts, as above.*

1. *If some lift of ψ acts with infinite order on $H_1(\widetilde{S}, \mathbb{C})$, then all lifts do.*
2. *If some lift $\widetilde{\psi}$ of ψ acts with $\rho(\widetilde{\psi}_*) > 1$ on $H_1(\widetilde{S}, \mathbb{C})$, then all lifts do. The spectral radii of all the lifts are the same.*
3. *The Nielsen–Thurston classification of ψ is the same as the classification of any of its lifts. The expansion factor of a lift of a pseudo-Anosov ψ agrees with the expansion factor on the base surface.*

Proof. The naturality of the map which associates a mapping class its collection of lifts implies that a power of a lift of a mapping class is a lift of a power of a mapping class. Thus, if we prove the proposition for some nonzero power of ψ, we have proved it for ψ. Let $\widetilde{\psi}$ be an arbitrary lift of ψ. Now consider a lift $g\widetilde{\psi}h$ of ψ and the square

$$(g\widetilde{\psi}h)^2 = g\widetilde{\psi}hg\widetilde{\psi}h.$$

We can slide the first h to the left, and the ψ–action permutes the cosets to give another double coset $gh'\widetilde{\psi}g\widetilde{\psi}h$. Similarly, we can slide the second g to the right to get $gh'\widetilde{\psi}^2g'h$. Since the Galois group is finite, some power of $\widetilde{\psi}$ induces the trivial permutation of the cosets. By the naturality of

the map which associates to a mapping class its lifts, we can assume that we started with a power of ψ which already trivially permuted the cosets, so that

$$(g\widetilde{\psi}h)^2 = gh\widetilde{\psi}^2 gh.$$

If the element gh has order n in the Galois group, then

$$(gh\widetilde{\psi}^2 gh)^n = (gh)^n \widetilde{\psi}^{2n}(gh)^n = \widetilde{\psi}^{2n}.$$

Thus, the action of $(g\widetilde{\psi}h)^{2n}$ coincides with that of $\widetilde{\psi}^{2n}$. The choice of g and h was arbitrary. Therefore, if one lift of ψ has infinite order as an automorphism of $H_1(\widetilde{S}, \mathbb{C})$ then each lift does. Similarly, if one lift has spectral radius λ then all lifts do.

Suppose that ψ has finite order, with order N. Then the lifts of ψ^N are the lifts of the identity, which are just the double coset $G \cdot 1 \cdot G$. Since G is finite, any element of the double coset has finite order. Suppose that ψ is reducible. Then ψ stabilizes a multicurve $\mathcal{C}$. If $\widetilde{\mathcal{C}}$ is the preimage of $\mathcal{C}$ in $\widetilde{S}$, observe that $\widetilde{\mathcal{C}}$ is a multicurve and that $\widetilde{\mathcal{C}}$ is $\widetilde{\psi}$–invariant for any lift of ψ. Finally suppose ψ is pseudo-Anosov with invariant foliations $\mathcal{F}^{\pm}$. Then each lift $\widetilde{\psi}$ stabilizes a pair of lifted foliations $\widetilde{\mathcal{F}}^{\pm}$, which are locally stretched and contracted respectively by the expansion factor λ. Each leaf of $\widetilde{\mathcal{F}}^{\pm}$ non–compact, since a closed leaf would descend to a closed leaf on S preserved by ψ. It follows that $\widetilde{\psi}$ is pseudo-Anosov. □

4. Growth of torsion homology

In this Section, we prove the results relating the surface conjectures and the torsion conjectures:

Theorem 4.1 (Theorem 1.5). *Let ψ be pseudo-Anosov and suppose that the mapping torus T_ψ has exponential torsion homology growth with respect to every exhausting tower of finite covers. Then there exists a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$.*

Theorem 4.2 (Theorem 1.6). *Let ψ be pseudo-Anosov and suppose that there exists a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$. Then there exists a tower of finite covers of the mapping torus T_ψ with exponential torsion homology growth.*

4.1. Cyclotomic fibrations.

Let

$$p : T_\psi \to S^1$$

be the fibration with fiber S and monodromy ψ. Suppose that for each lift $\widetilde{\psi}$ to a finite cover $\widetilde{S}$, we have $\rho(\widetilde{\psi}_*) = 1$. We call the pair (T_ψ, p) a **cyclotomic fibration**.

We would like to have a somewhat deeper understanding of cyclotomic fibrations, and we shall develop some of the theory here.

Suppose T_ψ is any mapping torus with fiber S. There is a natural map

$$\iota : \pi_1(S) \to \pi_1(T_\psi)$$

given by the inclusion of the fiber. Standard covering space theory implies the following characterization of the number of components of the preimage of S:

Lemma 4.3. *Let $X \to T_\psi$ be a finite Galois cover given by a map $\phi : \pi_1(T_\psi) \to G$ for some finite group G. The number of components of the preimage of S in X is equal to the index of the image of $\phi \circ \iota$ in G.*

Now one can give the following estimate about the growth of torsion homology for towers of covers of a cyclotomic fibration:

Lemma 4.4. *Let T_i be a degree d_i cover of T_ψ, and suppose that (T_ψ, p) is a cyclotomic fibration. Then for d_i sufficiently large, we have*

$$\frac{\log |H_1(T_i, \mathbb{Z})_{tors}|}{d_i} \leq \frac{C}{c_i} + O\left(\frac{1}{d_i}\right),$$

where C is a fixed constant and c_i is the number of components of the preimage of S in T_i.

Proof. Suppose that the fiber S of the fibration p is closed of genus $g > 1$. Then the absolute value Euler characteristic of S is $2g-2$. If $\widetilde{S}$ is a component of the preimage of S in M_i then $\widetilde{S}$ covers S with some degree q_i. The absolute value of the Euler characteristic of $\widetilde{S}$ is $q_i(2g-2)$, so that its genus is $q_i(g-1)+1$. By Lemma 4.3, we have that q_i divides d_i with quotient c_i, and c_i is the number of components of the preimage of S. The rank of the homology of $\widetilde{S}$ is $2q_i(g-1)+2$. By Lemma 3.9, we have that

$$\frac{\log |H_1(M_i, \mathbb{Z})_{tors}|}{d_i} \leq \frac{(\log 2)(2q_i(g-1)+2)}{q_i \cdot c_i} \leq \log 2 \frac{2(g-1)}{c_i} + \frac{2\log 2}{d_i}.$$

The argument for when S is not closed is analogous. □

Lemma 4.4 implies that if T_ψ admits a cyclotomic fibration then the torsion homology growth rate for any tower of covers of T_ψ is $O(1)$, and that it is $o(1)$ if the number of components of the preimage of the fiber of the cyclotomic fibration tends to infinity.

4.2. **Mapping tori admitting non–cyclotomic fibrations.** Now suppose there is a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$. Let $\widetilde{S}$ be the corresponding finite cover of S. There is a finite–by–infinite cyclic cover of T_ψ corresponding to the subgroup $\pi_1(\widetilde{S})$ of $\pi_1(T_\psi)$, and it is homeomorphic to $\widetilde{S} \times \mathbb{R}$. We have that $T_{\widetilde{\psi}}$ is a finite cover of T_ψ, and we write T_k for the finite cover $T_{\widetilde{\psi}^k}$ of $T_{\widetilde{\psi}}$.

Proposition 4.5. *Suppose that ψ admits a lift to a finite cover with $\rho(\widetilde{\psi}_*) > 1$. Then the tower of covers $\{T_k\}$ has exponential torsion homology growth.*

Proof. This follows from the fact that $\widetilde{\psi}_*$ has an eigenvalue of the unit circle. It follows that the torsion homology growth rate is given by the logarithm of the Mahler measure of $\widetilde{\psi}_*$, which is positive. The fact that this gives exponential torsion homology growth along the tower $\{T_k\}$ is essentially due to Silver and Williams in [19]. □

We immediately obtain the following:

Corollary 4.6 (Theorem 1.6)**.** *Suppose that T_ψ admits a non–cyclotomic fibration. Then there exists a tower of finite covers of T_ψ with exponential torsion homology growth.*

4.3. **A characterization of mapping tori admitting cyclotomic fibrations.** In this subsection we will show that if a mapping torus T_ψ admits one cyclotomic fibration then all fibrations of all finite covers of T_ψ are automatically cyclotomic as well. Let X be a finite CW complex. Recall that a **universal tower of abelian covers** of X is a tower $\{X_i\}$ of finite abelian covers of X with the property that if $X_A \to X$ is any finite abelian cover, the cover $X_i \to X$ factors through X_A for all sufficiently large i.

Theorem 4.7 (cf. Theorem 1.7)**.** *Let T_ψ be the mapping torus of a mapping class ψ. The following are equivalent:*

1. *The bundle T_ψ admits a fibration which is non–cyclotomic.*
2. *Each fibration of T_ψ is non–cyclotomic.*
3. *There exists a finite cover X of T_ψ such that some universal tower of abelian covers of X has exponential torsion homology growth.*

Proof. Clearly (2) implies (1). Suppose that T_ψ admits a cyclotomic fibration and that X is any finite cover of T_ψ. Fix a fiber S and a cyclotomic fibration monodromy $\widetilde{\psi}$ of X. In any universal tower of abelian covers of X, the number of components in the preimage of S tends to infinity. Indeed, in the homology cover X_N induced by $\pi_1(X) \to H_1(X, \mathbb{Z}/N\mathbb{Z})$, the number of preimages of the fiber is at least N. By definition, some level X_i of the universal tower of abelian covers will cover X_N, so that the number of preimages of the fiber in X_i is at least as large as it is in X_N. By Lemma 4.4, the torsion homology growth rate is $o(1)$ as N tends to infinity. Thus, (3) implies (2).

Now, suppose that T_ψ admits a non–cyclotomic fibration. Choose a lift $\widetilde{\psi}$ such that $\rho(\widetilde{\psi}) > 1$ on $H_1(\widetilde{S}, \mathbb{C})$. Choose a common finite cover of T_ψ and $T_{\widetilde{\psi}}$. We will abuse notation and denote the cover by T_ψ and the pre–chosen fiber by S. If $b_1(T_\psi) = 1$ then the exponential torsion homology growth with respect to any tower of homology covers follows from Proposition 4.5.

If $b_1(T_\psi) > 1$ then we have that the invariant cohomology $\mathrm{Hom}(\pi_1(S), \mathbb{Z})^\psi$ is nontrivial. Consider the homology covers $\{S_N\}$ of S which are ψ–invariant, namely those which arise from reduction of the ψ–coinvariant homology of S modulo N. The complex homology of each S_N splits into deck group eigenspaces, one for each irreducible representation χ of the deck group. The characteristic polynomial of ψ acting on these eigenspaces is given by $\chi(A)$, where A is the Alexander polynomial of T_ψ.

Consider the character variety

$$\mathfrak{X} = \mathrm{Hom}(\pi_1(S), S^1)^\psi.$$

Let ρ be the function on $\mathfrak{X}$ which assigns to a representation χ the modulus of the largest root of $\chi(A)$. Clearly ρ is a continuous function, which achieves at least one local maximum greater than one on $\mathfrak{X}$. It follows that as we range over the eigenvalues of $\widetilde{\psi}_*$ in its action on $H_1(S_N, \mathbb{C})$, for each $\lambda_0 \geq 1$ there is a proportion q such that at least q of the eigenvalues of $\widetilde{\psi}_*$ have modulus at least λ_0. If λ_0 is less than $\rho(\widetilde{\psi}_*)$ then q is positive.

Recall that if $A \in GL_n(\mathbb{Z})$ and Γ_A is the semidirect product obtained from $\mathbb{Z}$ and $\mathbb{Z}^n$ with the $\mathbb{Z}$–conjugation action given by A, then the torsion homology $H_1(\Gamma_A, \mathbb{Z})_{tors}$ is given by $\det{}'(A - I)$. We can now estimate the torsion homology growth of the homology covers of T_ψ. If $b_1(T_\psi) = n$ then the N^{th} homology cover T_N has degree approximately N^n over T_ψ. If T_ψ itself has no torsion homology then this estimate is precise. Otherwise, the estimate may be off by a constant factor which is no larger than the size of the torsion homology of T_ψ. The proportion of the eigenvalues of $\widetilde{\psi}_*$ acting on $H_1(S_N, \mathbb{C})$ with length at least λ_0 is $q > 0$. Note that for any complex number z, we have $|z - 1| \geq |z| - 1$. It follows that

$$|\det{}'(\widetilde{\psi}_*^N - I)| \geq (\lambda_0^N - 1)^{q \cdot (\mathrm{rk}\, H_1(S_N, \mathbb{Z}))} \cdot C_N,$$

where C_N is given by

$$\prod |\mu^N - 1|,$$

where μ ranges over the eigenvalues of $\widetilde{\psi}_*$ whose modulus is less than λ_0 and which are not equal to 1. The rank of $H_1(S_N, \mathbb{Z})$ is approximately proportional to the degree of S_N over S. Since the degree of S_N over S is N^{n-1}, we get two possibilities for the value of the rank, corresponding to whether of not S is closed. If S is closed of genus g then the genus g_N of S_N satisfies

$$(2g_N - 2) = N^{n-1}(2g - 2),$$

so that

$$\mathrm{rk}\, H_1(S_N, \mathbb{Z}) = N^{n-1}(2g - 2) + 2.$$

Otherwise, if r denotes the rank of $H_1(S, \mathbb{Z})$ then the rank r_N of $H_1(S_N, \mathbb{Z})$ satisfies

$$r_N - 1 = N^{n-1}(r - 1),$$

so that

$$\operatorname{rk} H_1(S_N, \mathbb{Z}) = N^{n-1}(r-1) + 1.$$

The logarithm of the right hand side of the inequality

$$|\det{}'(\widetilde{\psi}_*^N - I)| \geq (\lambda_0^N - 1)^{q \cdot (\operatorname{rk} H_1(S_N, \mathbb{Z}))} \cdot C_N$$

is bounded below by

$$|\chi(S)| \cdot q \cdot \log(\lambda_0^N - 1) \cdot N^{n-1} + \log C_N.$$

Thus the logarithm of the order of the torsion homology of T_N, normalized by the degree of the cover $T_N \to T_\psi$, is at least

$$\frac{|\chi(S)| \cdot q \cdot \log(\lambda_0^N - 1) \cdot N^{n-1} + \log C_N}{N^n} = |\chi(S)| \cdot q \cdot \frac{\log(\lambda_0^N - 1)}{N} + \frac{\log C_N}{N^n}.$$

Unfortunately, we have little control over the size of

$$\frac{\log C_N}{N^n},$$

and for a given N, this quantity may be very negative.

Replace $\widetilde{\psi^N}$ by $\widetilde{\psi^{kN}}$. In the limit as k tends to infinity, only the eigenvalues of norm at least one will contribute to the value of $\log C_{k \cdot N}/(k \cdot N^n)$, so that

$$\lim_{k \to \infty} \frac{\log C_{k \cdot N}}{k \cdot N^n} \geq 0.$$

This is because the eigenvalues occurring in

$$C_{k \cdot N} = \prod |\mu^{k \cdot N} - 1|$$

are the same as in the expansion for C_N, but their exponents are now larger.

Thus, we obtain

$$\lim_{k \to \infty} \left(|\chi(S)| \cdot q \cdot \frac{\log(\lambda_0^{k \cdot N} - 1)}{k \cdot N} + \frac{\log C_{k \cdot N}}{k \cdot N^n} \right) \geq |\chi(S)| \cdot q \cdot \log \lambda_0 > 0.$$

Thus, for each homology cover $X_N \to X$ we have produced a finite abelian cover $X_i \to X$ which covers X_N and which satisfies

$$\frac{\log |H_1(X_i, \mathbb{Z})_{tors}|}{\deg(X_i \to X)} \geq |\chi(S)| \cdot \frac{q}{2} \cdot \log \lambda_0.$$

Since for any finite abelian cover X_A of X there is a homology cover of X which covers X_A, it is clear that the desired universal tower of abelian covers exists. Therefore, (1) implies (3). □

We remark briefly on the role of the multiplicative constant $|\chi(S)|$ in the estimate above. On the one hand, the torsion homology growth exponent of a 3–manifold along towers of homology covers is independent of the choice of fibration. On the other hand, it is often possible to choose many different fibrations of a given hyperbolic 3–manifold, and the associated monodromies may have expansion factors tending to one and the Euler characteristics of the fibers tend to $-\infty$. Since the expansion factor provides an upper bound for λ_0, rescaling by $|\chi(S)|$ prevents our estimate from vanishing.

Corollary 1.9 follows more or less immediately now. Recall its statement:

Corollary 4.8. *Let ψ be a pseudo-Anosov mapping class. There exists a lift $\widetilde{\psi}$ to a finite cover satisfying $\rho(\widetilde{\psi}_*) > 1$ if and only if for each mapping class ϕ which is commensurable to ψ, there exists a lift $\widetilde{\phi}$ to a finite cover satisfying $\rho(\widetilde{\phi}_*) > 1$.*

Proof. Suppose ϕ is the monodromy of a bundle which is commensurable with T_ψ. By assumption there is a common finite cover X of T_ϕ and T_ψ which has exponential torsion homology growth with respect to some universal tower of abelian covers. It follows that ϕ cannot be the monodromy of a cyclotomic bundle. The converse is immediate. □

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